Advanced Mathematics Online: Assessing Particularities in the Online Delivery of a Second Linear Algebra Course

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Abstract

This article presents an overview of some issues that were confronted when delivering an online second Linear Algebra course (assuming a previous Introductory Linear Algebra course) to graduate students enrolled in a Secondary Mathematics Education program. The focus is on performance in one particular aspect of the course: *change of basis* and the *matrix representation of linear transformations with respect to different bases in Euclidean space*. A comparison is made with the performance of on-campus students. The methodology is qualitative, and situated within the onto-semiotic approach (Godino & Font, 2007). Language parameters are looked at, and the forum communication through written mathematics (both symbolic and mathematical English) amongst the online students is compared with in-class communication, where spoken mathematics and gestures play a larger role. It is emphasized that this article is exploratory, and tries to open possible avenues of research methodology about the online delivery of advanced mathematics. The topic was chosen with this exploratory objective in mind.

Introduction and Background

The present work is an overview of several key aspects and issues that were confronted when delivering an online second Linear Algebra course (assuming a standard Introductory Linear Algebra course as an antecedent) to graduate students enrolled in a Secondary Mathematics Education Masters program. All of the students were mathematics teachers at the high school level, as the online alternative was designed with this population in mind. The program itself is offered through the University System of Georgia (USG), and this course was offered through the Department of Mathematics and Statistics at Georgia State University (GSU). The course is cross listed in the usual catalogue, being a core course for undergraduate mathematics majors (4000 level) and a graduate course (6000 level) that must be passed by any student who, aspiring to enter to the graduate programs of Mathematics or Statistics, cannot prove mastery of the content that this course contains. The course is also frequently solicited by graduate students in Physics, Finance, and Economics, and it is required for graduate students in the program in Mathematics Education. The online course was only available for the students registered in the University System of Georgia program. This had been a preoccupation of the Department of Mathematics and Statistics as, in the beginning, it was thought that if the courses were offered online they might be available for students in general. However, it was made clear that only the students who were registered in the University System online program could take the online courses. This preoccupation, nevertheless, illustrated the existence of questions, doubts, and uncertainty in general in the Department of Mathematics and Statistics around the possibility of delivering a course of this nature in an online environment. This article will try to answer some of these questions, based on a first approximation to the problematic, pose alternative considerations based on this experience, and open up channels of discussion. It is important to mention that, while the population consisted of high school teachers, the course itself did not differ in the least from the course that is taught on campus, covering identical material from the same book (Leon, 2006), and there was no component geared specifically towards teachers. It was a content course and, as such, the issues that were raised will have to do with the delivery of this mathematical content in general.

The Course Content

The course consisted of fifteen weekly modules. The first week was a review of the subject matter that is covered in an Introductory Linear Algebra course, the fifth, tenth and fourteenth week consisted of reviews for the three online tests that were given, and the fifteenth week was a review for the final, which was proctored at different official sites throughout the state of Georgia. The content mirrored the sections covered in the book. Weeks two, three and four concentrated on Chapter four, ‘Linear Transformations’, as well as a return to Chapter three to review the study change of basis in Euclidean and abstract vector spaces. Weeks six, seven, eight and nine were dedicated to Chapter five, ‘Orthogonality’, with an emphasis on ‘Orthogonal Subspaces’, ‘Inner Product Spaces’, ‘Least Squares Problems’ (in Euclidean and abstract vector spaces), and ‘ Orthonormal sets’. During weeks eleven, twelve and thirteen the subjects covered were ‘Eigenvalues and Eigenvectors’, ‘Diagonalization’, ‘Hermitian Matrices’ and ‘Quadratic Forms’.

Some of the specific issues that came up, in terms of actual content, will be analyzed in later sections of the present article.

Delivery

The professor had taught the same course on campus several times, and actually taught it during the semester, simultaneously, of the online course delivery. For this reason, the course development was based on previous experience, familiarity with the book, knowledge of the specificities of the language and notation used (not always ‘standard’), as well as previous students’ issues with the material and the book.

The syllabus was available to the students as soon as they registered, and was very detailed in comparison to the syllabus used in the on campus course. The objectives of the course, the basic course contact information, exam dates, grading policy, and behavior policy were included in both, but the online course syllabus spelled out the details of the communication with the instructor, the logistics of the examination process, the weekly activities, technical support alternatives, and the testing center information for the final exam.

Each week the students were assigned specific reading in the Leon book, together with specific exercises. They were also required to participate in an online forum, in which they would get credit for being the first to solve an exercise, or explain it in an alternative way to their classmates. There was also an internet search component for the different topics, and a special page where the students posted their findings. The use of Matlab was a requirement, and exercises were assigned from the book and from the Atlast Computer Exercises (Leon, Herman & Faulkenberry, 1997) on a weekly basis. Each weekly module contained a list of online resources with the idea that the material, while not being mandatory, would complement the textbook. This material consisted of online texts, the MIT lectures on Linear Algebra, and assorted interactive websites and personal notes related to the different topics. It was emphasized that there were some differences in terminology and notation in the different texts and, as this is very common, even amongst high school algebra, geometry, precalculus and calculus textbooks, it was very good practice for real life issues in teaching mathematics at any level, or reading articles on mathematics and mathematics education. Finally, the professor had virtual office hours using the university’s Elluminate web conferencing system and, during the semester, several meetings with individual students and with groups were carried out. In particular there were sessions before each exam and before the final.

Flash-based presentations were made as a resource by the instructor. These consisted in the resolution of exercises for each section, many of which were theoretical in nature, but the presentations were not in lecture format. Only the written work was visible and, as it was being presented, it was narrated by the professor.

The communication between the instructor and the students, the posting of the presentations and other resources, as well as of the weekly modules, took place in Georgia Vista View, an instance of the Blackboard/WebCT learning management system (LMS). This was an enterprise-level system that served as a central LMS for several inter-institutional online degree programs, meaning students from different institutions across the state used a single LMS to access their courses online.

One of the big questions that faculty from the Department of Mathematics and Statistics unfamiliar with online delivery (including the professor) had at the beginning, was related to the how the syntax and specificities of mathematical language, both symbolic and embedded as part of ‘Mathematical English’ (Wells, 2003), would be dealt with. This actually did not result as problematic as foreseen, given that the students were comfortable with the “ASCII” conventions, and there was always the option of scanning written work and attaching it, or using an equation editor and attaching. The online tests were available for download on the indicated dates during the indicated hours, and the students scanned and uploaded their written answers. The professor graded in the usual way and returned the graded tests, with comments, as PDF files. The three semester exams were designed with the idea that students would be consulting their materials, but the final was cumulative, proctored, and students must have answered at least 70% correctly to pass the course, independently of their grades on the three exams, the Matlab assignments, and the forum and internet contributions.

Linear Algebra Learning

Several references to linear algebra learning are relevant to the present study (Dorier & Sierpinska, 2006; Alves-Dias, 2000; Dorier, 2000; Hillel, 2000; Dorier, Robert, Robinet & Rogalski, 1999; Sierpinska, 1999; Carlson Johnson, Lay, Porter, Watkins & Watkins, 1997). Hillel (2000), in particular, mentions that:

> By far, the most confusing case for students is … when the underlying vector space is already R^n. In this case an $n-$tuple is represented as another $n-$tuple relative to the basis β. Thus the object and its representation are the same 'animal', if not exactly the same list of values. Similarly, a matrix transformation A; first considered as a linear operator and then as having a (possibly different) matrix representation relative to a given basis (p. 201).

The question that will be analyzed in this study pertains, exactly, to this 'confusing' case. Hillel's study and the submitted paper (Montiel, Wilhelmi, Vidakovic & Elstak, 2009b) are the closest antecedents to the study on vectors, change of basis and matrix representations of a linear transformation on which the present analysis is based. What both studies agree upon is that "the persistence of mistakes with this kind of problem points to the existence of an obstacle that is of a more conceptual nature, and not just related to a difficulty in the operationalization of a procedure" (Hillel, 2000).

The group represented by Dorier (2000) has carried out research projects on the teaching and learning of linear algebra since the late 1980's. One of their conclusions is that the difficulties that students have with the formal aspects of linear algebra are content-specific. The conclusion that the difficulties are content-specific comes about after carrying out a historical analysis in which it is clear that the "…the basic idea of linear dependence was not so easy to formalize, even by great mathematicians like Euler" (Dorier, Robert, Robinet, Rogalski, 1999, 187). For the purposes of the present article, it is interesting to quote from Dorier and colleagues, that their "…teaching experiment pays much attention to changes in mathematical frameworks, semiotic registers of representation, languages or ways of thinking." (p. 105). In particular Alves-Dias (2000) reports that "Among the difficulties identified in linear algebra are: the number of new words to learn, comparable to a foreign language, (and)the totally new methods of exposition and demonstration…"

It should be evident from this section why the Mathematics and Statistics Department was concerned about the online delivery of a course of this nature.

Conceptual Framework

The goal of this study is to analyze the performance of the five graduate students of the online course in relation to the Linear Algebra topic of change of basis and matrix representation of a linear transformation with respect to different bases, and compare their performance with that of the on-campus class on the same exam question that both groups had to solve. The theoretical and methodological tools of the "Onto-semiotic Approach (OSA) (Godino, Batanero, Font, 2007), is used in the analysis.

In this section we offer a very brief notion of the framework.[1]

In OSA a mathematical object is anything that can be used, suggested or pointed to when doing, communicating or learning mathematics. OSA (Godino, Batanero & Roa, 2005; Font, Godino & D'Amore, 2007) considers six primary entities: *language* (terms, expressions, notations, graphics); *situations* (problems, extra or intra-mathematical applications, exercises, etc.); *definitions* or descriptions of mathematical notions (number, point, straight line, mean,function, etc.); *propositions*, properties or attributes, which usually are given as statements; *procedures* or subjects' actions when solving mathematical tasks (operations, algorithms, techniques, procedures); *and arguments* used to validate and explain the propositions or to contrast (justify or refute) subjects' actions.

The meaning of any mathematical object is the system of practices (operative and discursive) that a subject carries out to solve a certain type of problem in which the object is present. These types of correspondences, dependence relationships, or functions, between an antecedent (expression, representative, significant) and a consequence (content, significance), are established by some subject (person or institution) according to certain criteria or correspondence codes, and are named semiotic functions. These semiotic functions connect the objects amongst themselves and to the practices from which they originate.

By these means, the semiotic functions, and the associated mathematical ontology, take into account the essentially relational nature of mathematics and generalize, in a radical way, the notion of representation.

At the same time, in this particular study, the different registers available between an online course and a usual classroom setting are considered. The online course makes a much wider and richer use of written mathematical communication, especially in the forums, that were mandatory. The classroom ambience depends on the listening comprehension of spoken mathematics, and communication of written mathematics amongst the students (at least, communication to which the instructor had access)is negligible.

Analysis of Online and On-campus Performance

The question that will be presented was included on the final exam of the online course and the final exam of the on-campus course. The question will be shown (together with the expected answers), the responses will be presented quantitatively, and then a brief analysis of data, such as participation in the online forum, will be given.

Let $u_1 = (7,5),\ u_2 = (-3,-1)$
$v_1 = (1,-5),\ v_2 = (-2,2)$

and let **L** be a linear operator on R^2 whose matrix representation with respect to the ordered basis u_1, u_2 is

A= $\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$

a. Determine the transition matrix (change of basis matrix) from

V= $\{v_1, v_2\}$ to U= $\{u_1, u_2\}$. (Draw the commutative triangle)[2]

b. Find the matrix representation of **L** with respect to $\{v_1, v_2\}$.

Expected Answers

a. The transition matrix, using the technique of the commutative triangle and the notation specified is

$$S_U^V = V^{-1}U = -\frac{1}{8}\begin{pmatrix} 2 & 2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 7 & -3 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -5 & 2 \end{pmatrix}$$

b. The matrices A (which represents the linear transformation with respect to the basis U, and B, which represents the linear transformation with respect to the basis V are "similar". Then, if we call "S" the change of basis matrix that takes us from the basis U to the basis V (the one we calculated with the commutative diagram, "$V^{-1}U$")

and S^{-1} the matrix that takes us from the basis V to U, we have:

($B = SAS^{-1}$). $\begin{pmatrix} 35 & -21 \\ 58 & -35 \end{pmatrix}_V = \begin{pmatrix} -3 & 1 \\ -5 & 2 \end{pmatrix}^V_U \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}_U \begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}^U_V$

Of the five online students, three students answered parts a) and b) correctly, one student answered a) correctly and b) incorrectly, and one student answered both parts incorrectly. Of the 25 on-campus students, eleven students answered both parts correctly, nine students only part a) correctly and part b) incorrectly, and five answered both parts incorrectly.

The first aspect to consider is the number of students that answered part a) correctly and part b) incorrectly. It is noticeable that no student answered part b) correctly and part a) incorrectly. The students that answered both parts incorrectly, in all cases except two (on-campus students) in which the question was left in blank, tried to use the matrix representing the linear transformation with respect to the basis U. However, although this matrix was given as part of the general presentation of the question, it was not to be used until part b). The first part only asked for the students to find the change of basis matrix, which was to be used in the second part, together with the similarity relationship that exists between two matrices that represent the same linear transformation with respect to distinct bases.

It was in the on-campus context that more than a third of the students found the change of basis matrix, but could not apply the similarity relationship to find the matrix representing the linear transformation with respect to the basis V. Although both parts are operational in nature, students had been presented with the resource of the commutative triangle (an essential tool in modern mathematics) to find the change of basis matrix, and seemed to be able to ‘operate’. On the other hand, to find the matrix representing the linear transformation with respect to the basis V, it is necessary to keep track (‘bookkeeping’) of the logical sequence that changes the basis from V to U, carries out the linear transformation with respect to the basis U, and then changes the basis back from U to V. Although they had been presented with the similarity relation, and encouraged to use sub and super-indices, the conceptual aspect seems to have played a much greater role in the failure of correct operationalization in this task.

This study does not claim any quantitative validity. To begin with, the number of online students was much smaller than the on-campus group. However, through an analysis of the online students’ forum participation, it is feasible to classify some of the mathematical objects and semiotic functions that played a role in the system of practices that was developed, in some cases collectively.

The following student discussion was about this problem:
Given

$v_1 = (1,2),\ v_2 = (2,3),\ S = \begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix}$, find vectors w_1 and w_2

So that S will be the transition matrix from (v_1, v_2) to (w_1, w_2)

The five students are referred to as A, B, C, D and E. The notation of the following interchange is straight from their correspondence on the forum at WebCT. The example shown is between students A and B:

Student A: First we need to find U^-1 using the vectors v1 and v2.

U^-1 = [-3 2; 2 -1]

Now we are trying to find U^-1*[w1, w2] = S.

In order to find [w1, w2], I augmented U^-1 by S and performed row operations until U^-1 was the identity matrix.

This resulted in the matrix [5 1; 9 4] which corresponds to w1 and w2.

Therefore, w1 = (5, 9) and w2 = (1, 4).

Student B:

Hi **A**.
This is one of those situations I have been asking Dr. Montiel about, switching matrices to the other side of the equation. My approach was different; tell me what you think.
Since we are transitioning from w to v, we know v^-1*w=S. But the problem provided S, so I moved v^-1 to the other side. My new equation is w=vS. I achieved the same answer: [5 1;9 4] or w1=[5 1] and w2=[9,4].

oops, my error. w1=[5,9] and w2=[1 4]

Student A: I believe this should work as well. In fact, this give me a "Eureka" moment for question #8....which I had been working on for a while.
Question #8, to which Student A referred, was similar to the previous one, but had been causing him problems:

$v_1 = (2,6),\ v_2 = (1,4),\ S=\begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}$, find vectors
u_1 and u_2

So that S will be the transition matrix from
(v_1, v_2) to (u_1, u_2)

Student A: Since U^-1*v = S, we can rearrange this using the information given.

So, U^-1 = S*V^-1.

To find V^-1, I augmented matrix V by the identity.

V^-1 = [2 -1/2; -3 1]

Now we know S and V^-1, so U^-1 = S*V^-1 = [5 -1; 1 0].

But the question is asking for matrix U, not U^-1. This can be accomplished by finding the inverse of U^-1.

(U^-1)^-1 = U = [0 1; -1 5].

Therefore, u1 = (0, -1) and u2 = (1, 5).

*Special thanks to **B** for her inspiration in question #7. I had been working on this problem for a while, and her comment helped me figure this one out.

The systems of practices that can be identified in these interchanges are quite straightforward. This is different from the interview situation (Montiel, Wilhelmi, Vidakovic & Elstak, 2009a,b) in which it is much more difficult to isolate the expression and content of the semiotic functions, and the language is multilayered (for example, graphs and pictures, gestures, and so on play an important role in the interview situation as well as in the classroom, not only the algebraic and symbolic aspects as in the forum interchanges).

On the other hand, the communication and recognition of the work of colleagues is part of the history of the development of mathematics. In the past, many important ideas, discoveries and proofs were communicated through letters with colleagues. This aspect of mathematical communication is essential in the online environment, although the written aspect in the particular case that was analyzed is limited in comparison to the free hand of a written letter. It is important to mention that there were interchanges in which students scanned their work, and in those cases the language was more multifacetic.

Final Comments and Recommendations

It is inevitable that the online delivery of advanced mathematics courses will become more common as universities around the world develop and expand their virtual programs. While every program has its own characteristics, these final comments contain some recommendations in general that hopefully transcend GOML, and could be considered by other distance learning programs seeking similar implementations of advanced mathematics courses. The particularities of every subject, and the different levels of the subject, need to be considered as this form of higher education installs itself in the modern context. It is undeniable that, as a first generation, there is a certain reticence and many can feel uncomfortable with this type of delivery of the ideas and techniques of advanced mathematics. However, it is necessary to deal with this reality and, from the very beginning, look for ways to analyze and evaluate how this form of transmission of ideas can be enhanced, and how to exploit the features that differentiate it from the on-campus delivery. At the end, it might be an impetus to analyze, in general, the very complex subject of learning and teaching advanced mathematics. The present article is intent to open up this avenue of research.

The teaching and learning of advanced mathematical concepts is far from understood, and rigorous study of this phenomenon is relatively new. As different frameworks for carrying out these studies are being tested, technological development is bringing about forms of delivery that are entirely new and unchartered, for any academic endeavor. It is no wonder that uncertainty exists, especially when what was mentioned is coupled with a learning curve for those professors who will have to work with the new technology.

In the opinion of this author, it is undeniable that it can be done. One addition to the course described in this article has been the video-recording of all class lectures, available to students through iTunes U. The lectures can be downloaded, and care is taken to focus on the board work. This way, the online students have access to certain explanations, often spontaneous, that arise in the classroom setting, as well as other semiotic aspects, such as gestures, which are important in mathematics. It has been pointed out that spoken mathematics must reflect, in a linear manner, concepts that appear on the written page in a two dimensional format, such as superscripts and subscripts (Hayes, 1996); often the gestures of the professor are what fills in this discrepancy (Goldin-Meadow, Nusbaum, Kelly, & Wagner, 2001).

One important aspect, that cannot be overstated, is that online learning in advanced mathematics requires an enormous amount of self-discipline and motivation. While this is true in general for online delivery, in the particular case of this Linear Algebra course there is a constant connection between all the subjects, and each module serves as foundation for what comes next. It is also a second course in Linear Algebra, and the prerequisites must be specified and enforced, or there will be no way for the student to have success. The GOML students are often coming back to do graduate studies after many years and, while the majority had been mathematics majors and had taken an introductory Linear Algebra course at some point, many were very disconnected. More than half the students withdrew from the course. For this reason, the development of a preparatory course for this type of student is being considered, where they will be able to refresh their knowledge with aspects of introductory Linear Algebra, proof writing, Discrete Mathematics and Calculus. This course, if developed, would not count as towards the graduate degree, but would take care of this gap that many of the potential students confront, and would lead to the success, genuine understanding and mathematical growth that are goals of the advanced Linear Algebra course.

The chat room is an important tool for building community, although it should not be overused. Another addition to the previous course is the recording of all chat room review sessions, as well as the possibility of students meeting by themselves in groups in the chat room. However, although this possibility came about by suggestion of the students after the first review session, they have not actually taken advantage of the access they now have. This brings up the question, once again, of how much structure is needed to “force” the group work and other aspects of the online delivery that can be considered as advantageous in comparison to the in class situation. On the other hand, as I gain experience with the online delivery, I begin to incorporate some of these aspects into the on campus classes. The on campus students now have access to the flash presentations, the videos of the lectures and the possibility of virtual office hours. At the end, even within the traditional on campus context, the technology and methods developed for distance learning are going to embed themselves into the delivery of all subjects, advanced mathematics included, during the coming years. The concern is that this will not be in detriment to depth of understanding of sophisticated mathematical concepts. This article is an attempt to showcase possible ways in which this concern can be dealt with, and exemplify how the goal of theoretical understanding does not have to be compromised, as online learning by no means has to be mechanical. On the contrary, online resources can stimulate alternative methods of introducing the concepts, techniques and proofs of advanced mathematics (the many novel mathlets ‘out there’ on the internet during the past years is a concrete example), and there are means that can definitely be used to capture student engagement.

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Footnotes

1 The reader can find detailed presentations of these aspects in the cited articles, which can be found at: http://www.ugr.es/~jgodino/indice_eos.htm

2 The reference was to the following commutative triangle to help find the change of basis matrix from U to V.

$$U \xrightarrow{U} E$$

$S_U^V = V^{-1}U$ □ □ V^{-1}

V
